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Ultracomplete convergence vector spaces and the closed graph theorem

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1. Introduction

Closed graph theorems give sufficient conditions to guarantee that a linear map with a closed graph is continuous. Time has established this result as one of the fundamental principles of functional analysis. The first version was due to Banach [1] and took place in the setting of Fréchet spaces. This theorem proved to be so useful that great efforts were made over the next decades to increase its scope: to enlarge the classes of spaces which could act as domain spaces and codomain spaces for a closed graph theorem. One important result was that of Pták [10]. He extended the class of domain spaces to all barreled locally convex topological vector spaces and the codomain spaces to all B_r-complete spaces. This was a strong generalization of Banach's result but suffered two drawbacks. First, B_r-complete spaces have very poor permanence properties; second, they exclude some of the important spaces of functional analysis, for example, the distribution spaces.

Grothendieck [9] conjectured that, if the class of domain spaces for the closed graph theorem were restricted to Fréchet spaces, then the class of codomain spaces would have good permanence properties. In addition, a closed graph theorem for Fréchet spaces would easily give rise to one for ultrabornological spaces so that, if the class of codomain spaces were large enough, the closed graph theorem would encompass those locally convex topological vector spaces of importance in analysis. Slowikowski [13], Raikov [11] and De Wilde [6] all provided a positive answer to this conjecture (see [14]).

In this paper, we examine the closed graph theorem in a convergence vector space setting and give another solution to Grothendieck's conjecture. We exhibit a class of spaces, the

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ultracomplete convergence vector spaces which serve as an ideal class of codomain spaces for a closed graph theorem for Fréchet spaces. We show how the webs of De Wilde give rise to ultracomplete convergence vector spaces and how De Wilde's closed graph theorem sits very nicely in a convergence vector space setting.

2. Preliminaries

Let X be a set and suppose that to each x in X is associated a collection $\lambda(x)$ of filters on X satisfying:

(i) the ultrafilter $\dot{x} := \{A \subseteq X\colon\ x \in A\} \in \lambda(x)$,
(ii) if $\mathcal{F} \in \lambda(x)$ and $\mathcal{G} \in \lambda(x)$, then the filter $\mathcal{F} \cap \mathcal{G}$ also belongs to $\lambda(x)$,
(iii) if $\mathcal{F} \in \lambda(x)$ and $\mathcal{G} \supseteq \mathcal{F}$, then $\mathcal{G} \in \lambda(x)$.

The totality λ of filters $\lambda(x)$ for x in X is called a *convergence structure* for X, the pair (X, λ) a *convergence space* and filters $\mathcal{F}$ in $\lambda(x)$ *convergent* to x. We write $\mathcal{F} \to x$ instead of $\mathcal{F} \in \lambda(x)$.

A mapping $f: X \to Y$ between two convergence spaces X and Y is *continuous* if $f(\mathcal{F}) \to f(x)$ in Y whenever $\mathcal{F} \to x$ in X.

Let A be a subset of the convergence space X. The *adherence* $a(A)$ is the set of limit points of A, i.e., $a(A) = \{x \in X \mid \text{there is some filter } \mathcal{F} \to x \text{ in } X \text{ with } A \in \mathcal{F}\}$. A is *closed* if $A = a(A)$.

A convergence space X is *regular* if, whenever $\mathcal{F} \to x$ in X, the filter $a(\mathcal{F})$ generated by $\{a(P) \mid P \in \mathcal{F}\}$ also converges to x in X.

A convergence space X is *strongly first countable* if for each $x \in X$ there is a countable set $\mathcal{C}$ which is a local basis for X at x, i.e., for each $\mathcal{F} \to x$, there is a coarser $\mathcal{G} \to x$ with a basis from $\mathcal{C}$. For a topological space the notions of strong first countability and first countability coincide. As in topology, the strong first countability of a convergence space permits the use of sequences instead of filters in many arguments.

Let E be a vector space over the scalar field $\boldsymbol{K}$ of real or complex numbers and assume λ is a convergence structure on E. The pair (E, λ) is a *convergence vector space* if λ is compatible with the algebraic operations on E, i.e., if the mappings addition $+: E \times E \to E$ and scalar multiplication $\cdot: \boldsymbol{K} \times E \to E$ are continuous. In other words,

if $\mathcal{F} \to x$ and $\mathcal{G} \to y$ in E, then the filter $\mathcal{F} + \mathcal{G}$ generated by
$\{F + G\colon\ F \in \mathcal{F},\ G \in \mathcal{G}\}$ converges to $x + y$ in E,

and

if $\mathcal{F} \to x$ in E and $\mathcal{V} \to r$ in $\boldsymbol{K}$, then the filter $\mathcal{V} \cdot \mathcal{F}$ generated by the sets
$\{V \cdot F\colon\ V \in \mathcal{V},\ F \in \mathcal{F}\}$ converges to $r \cdot x$ in E.

A convergence vector space E is *locally convex* if for each filter which converges 0 in E there is a coarser filter with a basis of convex sets which still converges to 0. A filter $\mathcal{F}$ on E is a *Cauchy filter* if $\mathcal{F} - \mathcal{F}$ converges to 0 and E is *complete* if each Cauchy filter converges.

A convergence vector space need not be a topological vector space. We exhibit two important examples of convergence vector space structures on the dual $\mathcal{L}(E)$ of a locally convex topological vector space E. Neither structure is a topology in general.

Example 1.

(1) Let E be a locally convex topological vector space (or, more generally, a convergence vector space). Consider the evaluation mapping $\omega_E : \mathcal{L}(E) \times E \to \boldsymbol{K}$ given by $\omega_E(\varphi, x) = \varphi(x)$. We say $\Phi \to \varphi$ in $\mathcal{L}_c(E)$ if, whenever $\mathcal{A} \to x$ in E, $\omega_E(\Phi \times \mathcal{A}) \to \varphi(x)$ in $\boldsymbol{K}$. This is the *continuous convergence structure* on $\mathcal{L}(E)$ and $\mathcal{L}_c(E)$ carries the coarsest convergence structure making the evaluation continuous. It is well known [5] that $\mathcal{L}_c(E)$ is a locally compact convergence vector space if E is a topological vector space.

(2) A subset B of a convergence vector space E is called *bounded* if $\mathcal{N} \cdot B$ converges to 0, where $\mathcal{N}$ is the zero neighbourhood filter in $\boldsymbol{K}$. We define a convergence vector space structure μ on E as follows: $\Phi \to 0$ in (E, μ) if Φ contains a bounded set B and Φ is finer than $\mathcal{N} \cdot B$. Then $\mu(E) := (E, \mu)$ is the *Mackey* (or bornological) modification of E. Evidently the identity mapping $id : \mu(E) \to E$ is continuous.
When F is a locally convex topological vector space, the bounded sets of $\mathcal{L}_c(E)$ are the polars U^0 of zero neighborhoods U in E. The Mackey structure of $\mathcal{L}_c(E)$ is denoted $\mathcal{L}_\mu(E)$ and the convergent filters in $\mathcal{L}_\mu(E)$ are as usual called Mackey convergent.

Another important class of convergence vector spaces is given by inductive limits. Assume that (E_n) is an increasing sequence of convergence vector spaces such that the inclusion mappings $E_n \to E_{n+1}$ are all continuous. A convergence vector space E is called a *convergence inductive limit* of (E_n) if $E = \bigcup_n E_n$ and a filter $\mathcal{F}$ converges to x in E if and only if there is some n and a filter $\mathcal{G}$ on E_n such that $\mathcal{G}$ converges to x in E_n and $\mathcal{F}$ is finer than $[\mathcal{G}]$, the filter generated by $\mathcal{G}$ in E. We write $E = \operatorname{ind} E_n$. Note that E is in general not topological even if all E_n's are locally convex topological vector spaces and so the locally convex topological inductive limit $\operatorname{ind}_\tau E_n$ of the sequence (E_n) differs from the convergence inductive limit and we will call it the *topological inductive limit* for clarity. If (E_n) is a sequence of Fréchet spaces, E is called a convergence $\mathcal{LF}$-space.

For a more comprehensive treatment of convergence spaces and convergence vector spaces, the reader is referred to [5,7,8].

3. Ultracompleteness

Definition 2. Let E, F be convergence vector spaces and $u : E \to F$ a linear mapping. Then u is *nearly continuous* if for any filter $\mathcal{F} \to 0$ in E there is a filter $\mathcal{G} \to 0$ in F such that $a(u^{-1}(\mathcal{G})) \subseteq \mathcal{F}$.

If u is continuous one can set $\mathcal{G} = u(\mathcal{F})$ in the above definition and so continuous mappings are nearly continuous. Also, if E and F are topological vector spaces, u is nearly continuous if and only if $\overline{u^{-1}(\mathcal{U}_F(0))} \subseteq \mathcal{U}_E(0)$, i.e., if $\overline{u^{-1}(V)}$ is a zero neighbourhood in E for any zero neighbourhood V in F. This is just the classical definition of near continuity.

As in the classical case the proof of the closed graph theorem splits into two parts. The first is given by:

Proposition 3. *Let E be a Fréchet space, F a strongly first countable convergence vector space, $u: E \to F$ a linear mapping. Then u is nearly continuous.*

Proof. Let $\mathcal{U}$ be the zero neighbourhood filter in E. We have to show that there is some filter $\mathcal{G}_0 \to 0$ in F such that $\overline{u^{-1}(\mathcal{G}_0)} \subseteq \mathcal{U}$. Let $\mathcal{C}$ be a countable local basis of F at 0. Suppose the claim is false. Consider the family

$$\widetilde{\mathcal{C}} = \left\{C \in \mathcal{C} \mid \overline{u^{-1}(C - C)} \notin \mathcal{U}\right\}.$$

We show that each filter which converges in F to 0 contains an element of $\widetilde{\mathcal{C}}$. Suppose $\mathcal{G} \to 0$ in F. Then there is a filter $\mathcal{G}' \subseteq \mathcal{G}$ such that $\mathcal{G}' \to 0$ in F and $\mathcal{G}'$ has a basis of elements in $\mathcal{C}$. Since $\mathcal{G}' - \mathcal{G}' \to 0$, by assumption $\overline{u^{-1}(\mathcal{G}' - \mathcal{G}')} \nsubseteq \mathcal{U}$ and so there is some $G \in \mathcal{G}'$ with $\overline{u^{-1}(G - G)} \notin \mathcal{U}$. Since $\mathcal{G}'$ has a basis in $\mathcal{C}$, there is a $C \in \mathcal{C} \cap \mathcal{G}'$ with $C \subseteq G$ and hence $\overline{u^{-1}(C - C)} \notin \mathcal{U}$. Thus $C \in \mathcal{G}' \cap \widetilde{\mathcal{C}} \subseteq \mathcal{G} \cap \widetilde{\mathcal{C}}$ as required.

Now assume $y \in F$; then $(\frac{1}{i} y) \to 0$ in F and so there is a $C \in \widetilde{\mathcal{C}} \cap [(\frac{1}{i} y)]$, i.e., there is an i_0 such that $\{\frac{1}{i} y \mid i \geqslant i_0\} \subseteq C$. It follows that $y \in i_0 C$ and so

$$F = \bigcup \left\{iC \mid i \in \mathbb{N},\ C \in \widetilde{\mathcal{C}}\right\}.$$

Thus

$$E = u^{-1}(F) = \bigcup \left\{i u^{-1}(C) \mid i \in \mathbb{N},\ C \in \widetilde{\mathcal{C}}\right\}.$$

Since E is a Baire space, there are $i_0 \in \mathbb{N},\ C_0 \in \widetilde{\mathcal{C}},\ x_0 \in E$ and $U_0 \in \mathcal{U}$ such that

$$x_0 + U_0 \subseteq \overline{i_0 u^{-1}(C_0)}$$

and thus $(\frac{1}{i_0})x_0 + (\frac{1}{i_0})U_0 \subseteq \overline{u^{-1}(C_0)}$.

Let $\tilde{x} = (\frac{1}{i_0})x_0$ and $\widetilde{U} = (\frac{1}{i_0})U_0$; then $\tilde{x} + \widetilde{U} \subseteq \overline{u^{-1}(C_0)}$. Hence

$$\tilde{x} + \widetilde{U} - (\tilde{x} + \widetilde{U}) \subseteq \overline{u^{-1}(C_0)} - \overline{u^{-1}(C_0)} \subseteq \overline{u^{-1}(C_0) - u^{-1}(C_0)} \subseteq \overline{u^{-1}(C_0 - C_0)}.$$

It follows that $\overline{u^{-1}(C_0 - C_0)} \in \mathcal{U}$ contradicting the choice of C_0. $\square$

Definition 4. A convergence vector space E is called *ultracomplete* if E is strongly first countable and for every $\mathcal{F} \to 0$ in E, there is a countable subset $\{W_n\colon n \in \mathbb{N}\} \subseteq \mathcal{F}$ such that $[\{\sum_{k=n}^{\infty} W_k\colon n \in \mathbb{N}\}]$ converges to 0 in E.

A sequence (W_n) satisfying the condition of the definition is called a *rapid sequence.*

Remarks.

(1) Tacitly assumed in the definition above is that the filter elements are well-defined. Thus, if (W_n) is a rapid sequence and $x_n \in W_n$ for all n, then (x_n) is summable, i.e., the series $\sum_{n=1}^{\infty} x_n$ converges in E.

(2) If E is ultracomplete and the sequence (x_n) converges to 0 in E, then there is some subsequence (y_n) of (x_n) which is summable.

Proposition 5. *Let E be an ultracomplete convergence vector space. Then E is complete.*

Proof. Since E is strongly first countable, it suffices to show that every Cauchy sequence converges. Let (x_n) be a Cauchy sequence in E. If $T_k = \{x_n \mid n \geqslant k\}$, then $[\{x_m - x_n \mid m, n \geqslant k\}] = [T_k - T_k] \to 0$ in E. Since E is ultracomplete, we can find a sequence n_k of integers such that $(T_{n_k} - T_{n_k})$ is a rapid sequence. Choose a subsequence (y_k) of (x_n) such that $y_k \in T_{n_k}$ for all k. Then $\sum_{k=1}^{\infty}(y_k - y_{k+1})$ converges. But this series is $(y_1 - y_{k+1})$ and so (y_k) converges. Since (x_n) is Cauchy and has a convergent subsequence, it must converge. □

Example 6. Consider the convergence vector space $\mathcal{L}_c(l_2)$. Then this is a strongly first countable, complete space which is not ultracomplete. $\mathcal{L}_c(l_2)$ is strongly first countable since l_2 is a separable normed space [3]. Consider now the sequence $e_n = (0, 0, \ldots, 1, 0, 0, \ldots)$ (the 1 in the nth component) in $\mathcal{L}(l_2)$. The sequence is norm bounded and weak* convergent to 0. So $(e_n) \to 0$ in $\mathcal{L}_c(l_2)$. But there is no subsequence of (e_n) which is summable.

The ultracomplete spaces have excellent permanence properties:

Proposition 7. *Closed subspaces, quotients, countable products, countable coproducts and countable inductive limits of ultracomplete convergence vector spaces are ultracomplete. The countable projective limit of ultracomplete Hausdorff convergence vector spaces is ultracomplete.*

Proof. We first remark that all of the stated countable combinations of strongly first countable convergence vector spaces are strongly first countable (see [2]).

(i) Let E be ultracomplete and M a closed subspace. Suppose $\mathcal{F} \to 0$ in M. Then $[\mathcal{F}] \to 0$ in E so $[\mathcal{F}]$ contains a rapid sequence (W_n). Without loss of generality one can assume that $W_k \subseteq M$ for all k. Since M is a closed subspace of E the sets in the filter $[\sum_{k=n}^{\infty} W_k\colon\ n \in \mathbb{N}\}]$ make sense in M and the filter converges in M.

(ii) Let Q be a quotient of an ultracomplete convergence vector space E under the quotient mapping q. Let $\mathcal{G} \to 0$ in Q. Then there is a filter $\mathcal{F} \to 0$ in E such that $\mathcal{G} \supset q(\mathcal{F})$. Take a rapid sequence (W_k) in E then $(q(W_k))$ is a rapid sequence in F.

(iii) Let E be the inductive limit of the sequence (E_n) and denote by $e_n : E_n \to E$ the inclusion mappings. Let $\mathcal{F} \to 0$ in E. Then for some n, there is a filter $\mathcal{G} \to 0$ in E_n such that $e_n(\mathcal{G}) \subseteq \mathcal{F}$. If (W_k) is a rapid sequence in E_n then $(e_n(W_k))$ is a rapid sequence in E.

(iv) Let $E = \prod_{n=1}^{\infty} E_n$ be a countable product of ultracomplete convergence vector spaces and denote by $p_n : E \to E_n$ the projection mappings. Let $\mathcal{F} \to 0$ in E. Then each $p_n(\mathcal{F})$ converges to 0 and so it contains a rapid sequence $(W_{n,k})_k$. Set

$$\begin{aligned} W_1 &= W_{1,1} \times E_2 \times E_3 \times E_4 \times \cdots, \\ W_2 &= W_{2,1} \times W_{2,2} \times E_3 \times E_4 \times \cdots, \\ W_3 &= W_{3,1} \times W_{3,2} \times W_{3,3} \times E_4 \times \cdots \end{aligned}$$

and so on. Then $\{W_k\colon k \in \mathbb{N}\} \subseteq \mathcal{F}$. Also for each n

$$p_n\left(\left[\left\{\sum_{k=m}^{\infty} W_k\colon m \in \mathbb{N}\right\}\right]\right) \subseteq \left[\left\{\sum_{k=m}^{\infty} p_n(W_k)\colon m \in \mathbb{N}\right\}\right]$$
$$= \left[\left\{\sum_{k=m}^{\infty} (W_{n,k}\colon m \in \mathbb{N}\right\}\right]$$

and therefore $[\{\sum_{k=m}^{\infty}(W_k)\colon m \in \mathbb{N}\}]$ converges to 0 in $\prod_{n=1}^{\infty} E_n$.

(v) A countable projective limit of ultracomplete Hausdorff convergence vector space is ultracomplete since it is a closed subspace of a product of ultracomplete convergence vector spaces.

(vi) Since the direct sum is an inductive limit of finite products, the result holds for countable direct sums. □

Lemma 8. *Let E be a complete convergence vector space and (W_n) a basis for a filter $\mathcal{W} \subseteq \dot{0}$ on E which converges to 0 with the property that*

$$W_{n+1} + W_{n+1} \subseteq W_n \quad \text{for all } n \in \mathbb{N}.$$

Then, for all n, $\sum_{k=n}^{\infty} W_k \subseteq a(W_{n-1})$.

Proof. Choose a sequence (x_k) such that $x_k \in W_k$ for all $k \geqslant n$. Then, for all $r \geqslant n \geqslant 2$ we have

$$\begin{aligned} x_n + x_{n+1} + \cdots + x_r &\in W_n + W_{n+1} + \cdots + W_r \\ &\subseteq W_n + W_{n+1} + \cdots + W_r + W_r \\ &\subseteq W_n + W_{n+1} + \cdots + W_{r-1} + W_{r-1} \\ &\ \vdots \\ &\subseteq W_{n-1}. \end{aligned}$$

Now $\sum_{k=n}^{\infty} x_k$ is a Cauchy sequence since $\mathcal{W}$ converges to 0. By assumption E is complete, and so the sum converges with $\sum_{k=n}^{\infty} W_k \subseteq a(W_{n-1})$. □

Recall that a filter $\mathcal{F}$ in a convergence vector space E called *equable* if $\mathcal{N}\mathcal{F} = \mathcal{F}$. Also, E is called equable if for each filter which converges in E to 0 there is a coarser equable filter which still converges to 0. Every locally convex topological vector space is equable.

Proposition 9. *Let E be a strongly first countable, locally convex, equable, regular and complete convergence vector space. Then E is ultracomplete.*

Proof. We first show that for each filter $\mathcal{F}$ which converges to 0 in E there is a coarser filter $\mathcal{W}$ which still converges to 0 with a countable basis (W_n) such that $W_{n+1} + W_{n+1} \subseteq W_n$ for all n. Take a filter $\mathcal{F} \to 0$. Since E is locally convex and equable there is a filter

$\mathcal{G} \to 0$ in E such that $\mathcal{N}\mathcal{G} = \mathcal{G} \subseteq \mathcal{F}$. Since E is strongly first countable there is a filter $\mathcal{H} \subseteq \mathcal{G}$ with a countable basis such that $\mathcal{H} \to 0$ in E. Set $\mathcal{W} := co(\mathcal{N}\mathcal{H})$, then $\mathcal{W}$ has the desired property.

By the previous lemma, $[\{\sum_{k=n}^{\infty} W_k \colon n \in \mathbb{N}\}] \supset a(\mathcal{W})$. Since E is regular, $a(\mathcal{W}) \to 0$ in E and hence $[\{\sum_{k=n}^{\infty} W_k \colon n \in \mathbb{N}\}] \to 0$ as well. Thus E is ultracomplete. $\square$

Example 10.

(1) Every Fréchet space is ultracomplete. This is an immediate consequence of the preceding proposition.
(2) Every convergence $\mathcal{LF}$-space is ultracomplete. This is because countable inductive limits of ultracomplete convergence vector spaces are ultracomplete.
(3) If E is a Fréchet space then $\mathcal{L}_\mu(E)$ satisfies the conditions of the above proposition and hence is ultracomplete.

4. The closed graph theorem

Theorem 11. *Let E be a Fréchet space, F an ultracomplete convergence vector space and $u : E \to F$ a linear mapping with a closed graph. Then u is continuous.*

Proof. By Proposition 3, u is nearly continuous, and so there is a filter $\mathcal{G} \to 0$ in F such that $\overline{u^{-1}(\mathcal{G})} \subseteq \mathcal{U}(0)$. Choose a rapid sequence (G_n) in $\mathcal{G}$ such that $[\{\sum_{k=n}^{\infty} G_k \colon n \in \mathbb{N}\}]$ converges to 0 and a basis (U_n) of $\mathcal{U}(0)$ such that

$$U_n \subseteq \overline{u^{-1}(G_n)} \quad \text{for all } n \in \mathbb{N}.$$

Then

$$U_n \subseteq u^{-1}(G_n) + U_{n+1} \quad \text{for all } n \in \mathbb{N}$$

and we show that $u(U_n) \subseteq \sum_{k=n}^{\infty} G_k$ for all $n \in \mathbb{N}$. So let $z \in U_n$. Then there are $w_n \in u^{-1}(G_n)$ and $v_n \in U_n$ such that

$$\begin{aligned} z &= w_n + v_{n+1} \\ &= w_n + w_{n+1} + v_{n+2} \\ &\;\vdots \\ &= w_n + w_{n+1} + \cdots + w_{n+r} + v_{n+r+1}. \end{aligned}$$

Clearly $(\sum_{k=n}^{n+r} w_k)_r$ converges to z and $(u(\sum_{k=n}^{n+r} w_k))_r = (\sum_{k=n}^{n+r} u(w_k))_r$ converges also. Since the graph of u is closed,

$$u(z) = \lim_{r\to\infty} \sum_{k=n}^{n+r} u(w_k) = \sum_{k=n}^{\infty} u(w_k) \in \sum_{k=n}^{\infty} G_k.$$

Hence

$$u(U_n) \subseteq \sum_{k=n}^{\infty} G_k \quad \text{and} \quad u(\mathcal{U}) \supseteq \left[\left\{\sum_{k=n}^{\infty} G_k \colon n \in \mathbb{N}\right\}\right].$$

Since $[\{\sum_{k=n}^{\infty} G_k \colon n \in \mathbb{N}\}]$ converges to 0, so does $u(\mathcal{U})$. $\square$

Corollary 12 (Closed graph theorem). *Let E be a convergence inductive limit of Fréchet spaces, F an ultracomplete convergence vector space and $u : E \to F$ a linear mapping with a closed graph. Then u is continuous.*

Proof. Assume that E is the inductive limit of (E_i). Then the graph of $u \circ e_i$ is closed for all i and so $u \circ e_i$ is continuous for all i, giving the continuity of u. Here $e_i : E_i \to E$ denotes the inclusion mappings. $\square$

As usual, a closed graph theorem gives rise to a corresponding open mapping theorem.

Theorem 13 (Open mapping theorem). *Let E be an ultracomplete convergence vector space and F a convergence inductive limit of Fréchet spaces. Then every continuous linear mapping from E onto F is a quotient mapping.*

Proof. Denote by $q : E \to E/\ker(u)$ the quotient map and by $u_0 : E/\ker(u) \to F$ the associated map with $u_0 \circ q = u$. Then $u_0^{-1} : F \to E/\ker(u)$ is a linear mapping with a closed graph. By Proposition 7, $E/\ker(u)$ is ultracomplete and so u_0 is an isomorphism by Corollary 12. $\square$

The following sharpens the results in Theorem 11 considerably:

Proposition 14. *Let E be a Fréchet space and let F admit a* finer *convergence vector space structure which is ultracomplete. Any linear mapping $u : E \to F$ with a closed graph is continuous.*

Proof. Let F_0 denote the finer ultracomplete convergence vector space. Since $gr(u)$ closed in $E \times F$, it is closed in $E \times F_0$. Hence $u : E \to F_0$ is continuous and so also $u : E \to F$. $\square$

Corollary 15. *Let E and G be Fréchet spaces and $u : E \to \mathcal{L}_c(G)$ be a linear map with a closed graph. Then u is continuous.*

Proof. $\mathcal{L}_\mu(G)$ is ultracomplete and $id : \mathcal{L}_\mu(G) \to \mathcal{L}_c(G)$ is continuous. $\square$

The following are some simple consequences of the above results.

Proposition 16. *Let E be a Fréchet space. There is no* strictly finer *convergence vector space structure λ making E_λ an ultracomplete convergence vector space.*

Proof. If there were, then the identity mapping $E_\lambda \to E$ would be continuous and hence an isomorphism by the open mapping theorem. □

Corollary 17. *Let E be a convergence $\mathcal{LF}$-space and assume that F and G are two algebraically complementary closed subspaces of E. Then E is the topological sum of F and G, i.e., $E = F \bigoplus G$.*

Proof. A common argument shows that the projection $q : E \to F$ has a closed graph. As a closed subspace of an ultracomplete space F is itself ultracomplete and so q is continuous by Corollary 12. □

The previous results can be used to derive some important locally convex topological vector space results.

First let us recall that, to any convergence vector space E, one can associate a locally convex topological vector space $\tau(E)$, the finest locally convex topological vector space coarser than E. The correspondence $E \to \tau(E)$ is a reflector. When $E = \operatorname{ind} E_i$ is a convergence inductive limit, $\tau(E)$ is the topological inductive limit of the sequence $(\tau(E_i))$, i.e., $\tau(E) = \operatorname{ind}_\tau E_i$.

Corollary 18. *Let E be a convergence inductive limit of Fréchet spaces, F an ultracomplete convergence vector space and $u : E \to F$ a linear mapping. Then u is continuous if and only if $u : \tau(E) \to \tau(F)$ is continuous.*

Proof. It is well known that, if $u : E \to F$ is continuous then $u : \tau(E) \to \tau(F)$ is continuous. If $u : \tau(E) \to \tau(F)$ is continuous, its graph is closed in $\tau(E) \times \tau(F)$ and therefore in $E \times F$. Consequently, $u : E \to F$ has a closed graph and is continuous by Corollary 12. □

Corollary 18 applies in particular if E and F are topological $\mathcal{LF}$-spaces:

Corollary 19. *Let $E = \operatorname{ind}_\tau(E_n)$ and $F = \operatorname{ind}_\tau(F_n)$ be two topological $\mathcal{LF}$-spaces and $u : E \to F$ be a continuous map. Then for all m there is an n such that $u(E_m) \subseteq F_n$ and $u : E_m \to F_n$ is continuous.*

Proof. From Proposition 18, we get that $u : \operatorname{ind} E_n \to \operatorname{ind} F_n$ is continuous. If $\mathcal{U}_m$ is the zero neighbourhood filter in E_m, then $u(\mathcal{U}_m)$ converges to 0 in $\operatorname{ind} F_n$ and so there is an n such that $u(\mathcal{U}_m)$ converges to 0 in F_n. So $u(E_m) \subseteq F_n$ and $u : E_m \to F_n$ is continuous. □

From this we get Grothendieck's factorization theorem (see, e.g., [4]).

Theorem 20. *Let E be a Fréchet space, $F = \operatorname{ind}_\tau F_n$ a topological $\mathcal{LF}$-space and $u : E \to F$ a continuous map. Then, there is some $n \in \mathbb{N}$ such that $u(E) \subseteq F_n$ and $u : E \to F_n$ is continuous.*

Theorem 21. *Let E and F be locally convex topological vector spaces, E ultrabornological and F a topological $\mathcal{LF}$-space. Then every linear mapping $u : E \to F$ with a closed graph is continuous.*

Proof. Let $\mu(E)$ be the Mackey modification of E and set $F_\lambda = \operatorname{ind} F_n$. Then $\tau(\mu(E)) = E$ and $\tau(F_\lambda) = F$. Furthermore $u : \mu(E) \to F_\lambda$ is a linear mapping from a convergence inductive limit of Banach spaces to an ultracomplete convergence vector space. Since $gr(f)$ is closed in $E \times F$, it is closed in $\mu(E) \times F_\lambda$. By Corollary 12, $u : \mu(E) \to F_\lambda$ is continuous and so also $u : E \to F$. $\square$

5. Webs and web-spaces

To conclude this report, we show how the closed graph and open mapping theorems of De Wilde for locally convex topological vector spaces fit into the the previous results.

Let I be the set of all finite sequences of natural numbers and F be a vector space. A family $\mathcal{W} = \{W_\alpha \colon \alpha \in I\}$ is called a *web* on F if the following hold:

(i) $\bigcup\{W_i \colon i \in \mathbb{N}\}$ absorbs E.

(ii) If $\alpha \in I$ then $\bigcup\{W_{(\alpha,i)} \colon i \in \mathbb{N}\}$ absorbs W_α.

(iii) $W_{(\alpha,i)} + W_{(\alpha,i)} \subseteq W_\alpha$ for all $\alpha \in I$ and all $i \in \mathbb{N}$.

A sequence $\mathcal{S} = (S_n)$ of subsets of E is called a *strand* if there is a sequence (k_n) in $\mathbb{N}$ such that $S_n = W_{k_1,\dots,k_n}$ for all $n \in \mathbb{N}$.

The strands are filter bases and give rise to a convergence space. Slight assumptions on the web [2] make this a convergence vector space which we call the *web-space* and denote by $F_{\mathcal{W}}$. A convergence vector space F is said to be webbed (by $\mathcal{W}$) if the strands of $F_{\mathcal{W}}$ converge in F, i.e., if the identity mapping $F_{\mathcal{W}} \to F$ is continuous.

Finally, a web is called *tight* [12] if, for each strand $\mathcal{S} = (S_k)$ and each sequence (x_k) with $x_k \in S_k$ for all k, the series $\sum_{k=1}^{\infty} x_k$ converges in F and $\sum_{r=k+1}^{\infty} x_r \in S_k$ for all k.

The connection between tight web and the convergence theory is now given by:

Proposition 22. *Let F be a convergence vector space webbed by $\mathcal{W}$. If the web $\mathcal{W}$ is tight, then the web space $F_{\mathcal{W}}$ is ultracomplete.*

Proof. Since $\mathcal{W}$ is a countable local base of $F_{\mathcal{W}}$ at 0, the space $F_{\mathcal{W}}$ is strongly first countable. If $\mathcal{F} \to 0$ in $F_{\mathcal{W}}$ then there is a strand $\mathcal{S} = (S_n)$ such that $\mathcal{F} \supseteq [\mathcal{S}]$. If now (x_n) is a sequence in E such that $x_n \in S_n$ for all n then $\sum_{k=n+1}^{\infty} x_n \in S_n$ for all n and so

$$\left[\left\{ \sum_{k=n+1}^{\infty} x_n \colon n \in \mathbb{N} \right\}\right] \supseteq [\mathcal{S}].$$

Therefore $\sum_{k=1}^{\infty} x_n$ actually converges in $F_{\mathcal{W}}$ and so the filter $[\sum_{k=n+1}^{\infty} S_n\}]$ is well-defined in $F_{\mathcal{W}}$ and finer than $[\mathcal{S}]$, which means that it converges in $F_{\mathcal{W}}$. $\square$

Proposition 22 now allows a direct proof of the following well known theorem (see, e.g., [12, Theorem 16]):

Theorem 23. *Let E be a Fréchet space, F a convergence vector space with a tight web, $u : E \to F$ a linear mapping with a closed graph. Then u is continuous.*

Proof. The web-space $F_{\mathcal{W}}$ is ultracomplete. By Theorem 11, $u : E \to F_{\mathcal{W}}$ is continuous, hence also $u : E \to F$. □

It is important to note here that the continuity of u is actually into $F_{\mathcal{W}}$, occasionally a much sharper result than continuity into F.

References

[1] S. Banach, Théorie des Opérations Linéaires, Chelsea, New York, 1978 (reprint from 1932).
[2] R. Beattie, H.-P. Butzmann, Strongly first countable convergence spaces, in: Convergence Structures 1984, Akademie, Berlin, 1985, pp. 39–46.
[3] R. Beattie, H.-P. Butzmann, Sequentially determined convergence spaces, Czechoslovak Math. J. 37 (1987) 231–247.
[4] K.-D. Bierstedt, An introduction to locally convex inductive limits, in: Functional Analysis and its Applications, Nice, 1986, ICPAM Lecture Notes, World Scientific, Singapore, 1988.
[5] E. Binz, Continuous Convergence in $\mathcal{C}(X)$, Lecture Notes in Math., Vol. 469, Springer, Berlin, 1975.
[6] M. De Wilde, Réseaux dans les espaces linéaires à semi-normes, Mém. Soc. R. Sci. Liège 18 (1969) 1–144.
[7] W. Gähler, Grundstrukturen der Analysis, I, Birkhäuser, Basel, 1977.
[8] W. Gähler, Grundstrukturen der Analysis, II, Birkhäuser, Basel, 1978.
[9] A. Grothendieck, Produits tensoriels topologiques et espaces nucléaires, Mem. Amer. Math. Soc. 16 (1955).
[10] V. Pták, Completeness and the open mapping theorem, Bull. Soc. Math. France 86 (1958) 41–74.
[11] D.A. Raikov, Double closed graph theorem for topological linear spaces, Siberian Math. J. 7 (1966) 287–300.
[12] W. Robertson, On the closed graph theorem and spaces with webs, Proc. London Math. Soc. 24 (1972) 692–738.
[13] W. Slowikowski, On continuity of inverse operators, Bull. Amer. Math. Soc. 67 (1961) 467–470.
[14] M. Valdivia, On Slowikowski, Raikov and De Wilde Closed Graph Theorems, in: J.A. Barroso (Ed.), Aspects of Mathematics and its Applications, Elsevier Science, Amsterdam, 1986.